B.Sc Part II

Infinite Series 19-11-20

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Cauchy's Condensation Test – If the function $f(n)$ is positive for all positive integral values of n and continually decreases as n increases and if a be any positive integer then the two infinite series
$f(1) + f(2) + f(3) + \cdots$ i.e.
$\Sigma f(n)$ and $a f(a) + a^2 f(a^2) + a^3 f(a^3) + \cdots$ i.e.
$\Sigma a^n f(a^n)$ are both convergent or both divergent.

Proof: The terms in the series $\Sigma f(n)$ can be arranged as follows:

$$\{f(1) + f(2) + f(3) + \cdots + f(a)\}$$
$$+ \{f(a+1) + f(a+2) + \cdots + f(a^2)\}$$
$$+ \{f(a^2+1) + f(a^2+2) + \cdots + f(a^3)\}$$
$$+ \cdots$$
$$+ \{f(a^k+1) + f(a^k+2) + \cdots + f(a^{k+1})\}$$
$$+ \cdots$$

Now, in the first group of terms the number of terms in a, in the

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Second group of terms the number of terms is (a^2-a), in the third group of terms the number of terms is (a^3-a^2) and so on.
Consider the $(k+1)^{th}$ group of terms viz;

$f(a^k+1)+f(a^k+2)+\dots+f(a^{k+1})$

In this group; the number of terms $= a^{k+1}-a^k = a^k(a-1)$

Also, since the function $f(n)$ continually decreases as n increases.
Hence in this group of terms every term is greater than $f(a^{k+1})$
Therefore

$$f(a^k+1)+f(a^k+2)+\dots+f(a^{k+1})$$
$$> a^k(a-1)f(a^{k+1}) = \frac{a-1}{a}\cdot a^{k+1}f(a^{k+1})$$

Putting $k=0,1,2,3,\dots$ successively we get

$$f(2)+f(3)+\dots+f(a) > \frac{a-1}{a}\{af(a)\}$$

when $k=0$

$$\{f(a+1)+f(a+2)+\dots+f(a^2)\} > \frac{a-1}{a}\{a^2f(a^2)\}$$ when $k=1$

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$\{f(a^2+1)+f(a^2+2)+ \cdots +f(a^3)\} >$

$\frac{a-1}{a}\{a^3 f(a^3)\}$ when $K=2$

Adding, we get

$$\{\Sigma f(n) - f(1)\} > \frac{a-1}{a} \cdot \Sigma a^n f(a^n)\}$$

Hence by the comparison test, if the series $\Sigma a^n f(a^n)$ is divergent then $\Sigma f(n)$ is also divergent.

Again, the number of terms in the $(K+1)^{th}$ group is $(a-1)a^K$ and each term in this group is less than than $f(a^K)$.

Hence we have

$$f(a^K+1)+f(a^K+2)+ \cdots +f(a^{K+1})$$

$$< (a-1)\, a^K f(a^K)$$

Putting $K=0,1,2,3, \ldots$ successively we get

$f(2)+f(3)+ \cdots +f(a) < a-1\{f(1)\}$

when $K=0$

$f(a+1)+f(a+2)+ \cdots + f(a^2) <$

$(a-1)\{a f(a)\}$, when $k=1$

$f(a^2+1)+f(a^2+2)+ \cdots + f(a^3) <$

$(a-1)\{a^2 f(a^2)\}$ when $k=2$

Adding, we get

$\Sigma f(n) - f(1) < (a-1)\{f(1)+\Sigma a^n f(a^n)\}$

Hence if $\Sigma a^n f(a^n)$ is convergent, then $\Sigma f(n)$ is also convergent.